ON THE DETERMINATION OF STRESS CONCENTRATIONS ON THE BASIS OF THE APPLIED THEORY

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In the present paper the possibility is investigated of using the equations of the applied theory of bending plates to calculate stress concentrations. The asymptotic expansion of the solution of the corresponding problems of elasticity theory obtained in [1] is utilized. It is shown that in a number of cases the calculation of stress concentrations on the basis of the applied theory is possible.

1. We consider a thin plate of thickness $2h$, bounded by a cylindrical surface Γ_1. We allow the plate to have a hole bounded by the cylindrical surface Γ_2 (Fig.1). We shall assume that the distance between Γ_1 and Γ_2 is sufficiently large in comparison with the thickness of the plate. At the same time we assume that the diameter a of the hole is also considerably larger than the thickness of the plate. The surface Γ_1 of the plate is loaded by some system of forces which is statically equivalent to zero, while the plane surfaces Γ_3 of the plate are stress-free. In this case, as is known, a stress concentration occurs at Γ_2.

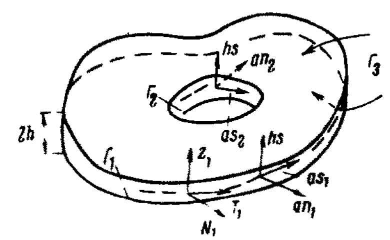

Fig. 1

The methods of calculating the stress concentrations in such problems on the basis of the applied theory have been treated in a number of works. A summary of the results may be found in [2].

The basic purpose of the present paper is to elucidate the connection between the relations of the applied theory and the exact solution of the corresponding problem in elasticity theory.

We note that by the applied theory of bending of a plate we mean the theory based on the relationships

$$\sigma_x = -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right), \quad \sigma_y = -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right), \quad \tau_{xy} = -\frac{Ez}{1+\sigma}\frac{\partial^2 w}{\partial x \partial y}$$

$$\tau_{xz} = -\frac{E(h^2 - z^2)}{2(1-\sigma^2)}\frac{\partial}{\partial x}\Delta w, \qquad \tau_{yz} = -\frac{E(h^2 - z^2)}{2(1-\sigma^2)}\frac{\partial}{\partial y}\Delta w, \qquad \sigma_z = 0 \tag{1.1}$$

where $w(x,y)$ is the displacement of points of the middle surface of the plate, which is a biharmonic function determined by the Kirchhoff boundary conditions, E is the elastic modulus, σ is Poisson's ratio, and Δ is the Laplace operator.

2. We introduce, as in [1] dimensionless coordinates s_1, n_1 and ζ (Fig.1) connected with Γ_1. We denote the forces acting on Γ_1 by $N_1(s_1, \zeta)$, $T_1(s_1, \zeta)$, $Z_1(s_1, \zeta)$. In [1] it was shown that the state of stress of a solid plate bounded by the surface Γ_1 and subjected to the aforementioned forces may be represented in the following form for sufficiently small $\lambda = h/a$:

$$\sigma_{n_1}^{\circ} = 2\mu\lambda \left\{ \left[2\nu \frac{\partial^2\psi}{\partial n_1^2} + (\nu - 1)\left(\frac{1}{H_1^2}\frac{\partial^2\psi}{\partial s_1^2} + \frac{1}{H_1}\frac{a}{R_1}\frac{\partial\psi}{\partial n_1} + \frac{1}{H_1^3} n_1 \frac{aR_1'}{R_1^2}\frac{\partial\psi}{\partial s_1}\right)\right]\zeta - \right.$$
$$- \left(\nu + \frac{1}{3}\right)\lambda^2 \frac{\partial^2\Delta\psi}{\partial n_1^2}\frac{\zeta^3}{2}\right\} + 2\mu\lambda \sum_{p=1}^{\infty} [(\nu-1) s_p(\zeta) + \gamma_p^2 n_p(\zeta)] \times$$
$$\times \left(1 - n_1\frac{a}{2R_1} + n_1^2 \frac{3a^2}{8R_1^2} - \dots\right) c_{p2}(s_1) \times$$
$$\times \exp\frac{\gamma_p n_1}{\lambda} + 2\mu\lambda^2 \left\{ 2\sum_{k=0}^{\infty} \sin\sigma_k\zeta \frac{1}{H_1}\left[b_{k1}(s_1)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \dots\right)\right]_{s_1}' \times \right.$$
$$\times \exp\frac{\sigma_k n_1}{\lambda} + (\nu - 1)\sum_{p=1}^{\infty} s_p(\zeta)\left[\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \dots\right) c_{p3}(s_1) + \right.$$
$$+ \frac{1}{2\gamma_p}\left(-n_1\frac{a^2}{4R_1^2} - n_1\frac{\partial^2}{\partial s_1^2} + \dots\right) c_{p2}(s_1)\right]\exp\frac{\gamma_p n_1}{\lambda} + \sum_{p=1}^{\infty} n_p(\zeta)\gamma_p\left[\gamma_p\left(1 - n_1\frac{a}{2R_1} + \right.\right.$$
$$\left.\left. + n_1^2\frac{3a^2}{8R_1^2} - \dots\right) c_{p3}(s_1) + \left(-\frac{a}{R_1} + n_1\frac{11a^2}{8R_1^2} - n_1\frac{1}{2}\frac{\partial^2}{\partial s_1^2} + \dots\right) c_{p2}(s_1)\right]\right\} \exp\frac{\gamma_p n_1}{\lambda} + \dots \tag{2.1}$$

$$\sigma_{s_1}^{\circ} = 2\mu\lambda \left\{ \left[2\nu\left(\frac{1}{H_1^2}\frac{\partial^2\psi}{\partial s_1^2} + \frac{1}{H_1}\frac{a}{R_1}\frac{\partial\psi}{\partial n_1} + \frac{1}{H_1^3}n_1\frac{aR_1'}{R_1^2}\frac{\partial\psi}{\partial s_1}\right) + (\nu - 1)\frac{\partial^2\psi}{\partial n_1^2}\right]\zeta - \right.$$
$$\left. - \left(\nu + \frac{1}{3}\right)\lambda^2\left(\frac{1}{H_1^2}\frac{\partial^2\Delta\psi}{\partial s_1^2} + \frac{1}{H_1}\frac{a}{R_1}\frac{\partial\Delta\psi}{\partial n_1} + \frac{1}{H_1^3}n_1\frac{aR_1'}{R_1^2}\frac{\partial\Delta\psi}{\partial s_1}\right)\frac{\zeta^3}{2}\right\} +$$
$$+ 2\mu\lambda(\nu - 1)\sum_{p=1}^{\infty} s_p(\zeta)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} + \dots\right) c_{p2}(s_1)\exp\frac{\gamma_p n_1}{\lambda} +$$
$$+ 2\mu\lambda^2\left\{ -2\sum_{k=0}^{\infty}\sin\sigma_k\zeta\frac{1}{H_1}\left[b_{k1}(s_1)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \dots\right)\right]_{s_1}'\exp\frac{\sigma_k n_1}{\lambda} + \right.$$
$$+ (\nu - 1)\sum_{p=1}^{\infty} s_p(\zeta)\left[\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \dots\right) c_{p3}(s_1) + \right.$$
$$\left. + \frac{1}{2\gamma_p}\left(-n_1\frac{a^2}{4R_1^2} - n_1\frac{\partial^2}{\partial s_1^2} + \dots\right) c_{p2}(s_1)\right]\exp\frac{\gamma_p n_1}{\lambda} + \tag{2.2}$$
$$\left. + \frac{1}{H_1}\frac{a}{R_1}\sum_{p=1}^{\infty} n_p(\zeta)\gamma_p\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \dots\right) c_{p2}(s_1)\exp\frac{\gamma_p n_1}{\lambda}\right\} + \dots$$

$$\tau^\circ_{n_1s_1} = 2\mu\lambda \left\{(\nu+1)\,\zeta\left(\frac{1}{H_1}\frac{\partial^2\psi}{\partial n_1\partial s_1} - \frac{1}{H_1^2}\frac{a}{R_1}\frac{\partial\psi}{\partial s_1}\right) - \left(\nu+\frac{1}{3}\right)\lambda^2\frac{\zeta^3}{2}\left(\frac{1}{H_1}\frac{\partial^2\triangle\psi}{\partial n_1\partial s_1} - \right.\right.$$

$$\left.\left. - \frac{1}{H_1^2}\frac{a}{R_1}\frac{\partial\triangle\psi}{\partial s_1}\right)\right\} + 2\mu\lambda\left\{-\sum_{k=0}^{\infty}\sigma_k\sin\sigma_k\zeta\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)b_{k1}(s_1)\exp\frac{\sigma_k n_1}{\lambda}\right\} +$$

$$+ 2\mu\lambda^2\left\{\sum_{k=0}^{\infty}\sin\sigma_k\zeta\left[\frac{1}{H_1}\frac{a}{R_1}\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)b_{k1}(s_1) - \sigma_k\left(1 - n_1\frac{a}{2R_1} + \right.\right.\right.$$

$$\left. + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)b_{k2}(s_1) - \left(-\frac{a}{R_1} + n_1\frac{11a^2}{8R_1^2} - n_1\frac{1}{2}\frac{\partial^2}{\partial s_1^2} + \ldots\right)b_{k1}(s_1)\bigg]\exp\frac{\sigma_k n_1}{\lambda} +$$

$$\left. + \frac{1}{H_1}\sum_{p=1}^{\infty}\gamma_p n_p(\zeta)\left[c_{p2}(s_1)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)\right]'_{s_1}\exp\frac{\gamma_p n_1}{\lambda}\right\} + \ldots \qquad (2.3)$$

$$\tau^\circ_{n_1z} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial\triangle\psi}{\partial n_1} + 2\mu\lambda\sum_{p=1}^{\infty}\gamma_p r_p(\zeta)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)c_{p2}(s_1)\exp\frac{\gamma_p n_1}{\lambda} +$$

$$+ 2\mu\lambda^2\left\{\sum_{k=0}^{\infty}\frac{1}{H_1}\cos\sigma_k\zeta\left[b_{k1}(s_1)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)\right]'_{s_1}\exp\frac{\sigma_k n_1}{\lambda} + \right.$$

$$+ \sum_{p=1}^{\infty} r_p(\zeta)\left[\gamma_p\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)c_{p3}(s_1) + \right.$$

$$\left.\left. + \left(-\frac{a}{2R_1} + n_1\frac{5a^2}{8R_1^2} - n_1\frac{1}{2}\frac{\partial^2}{\partial s_1^2} + \ldots\right)c_{p2}(s_1)\right]\exp\frac{\gamma_p n_1}{\lambda}\right\} + \ldots \qquad (2.4)$$

$$\tau^\circ_{s_1z} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{1}{H_1}\frac{\partial\Delta\psi}{\partial s_1} - 2\mu\lambda\sum_{k=0}^{\infty}\sigma_k\cos\sigma_k\zeta\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)\times$$

$$\times\, b_{k1}(s_1)\exp\frac{\sigma_k n_1}{\lambda} + 2\mu\lambda^2\left\{-\sum_{k=0}^{\infty}\cos\sigma_k\zeta\left[\left(-\frac{a}{2R_1} + n_1\frac{5a^2}{8R_1^2} - n_1\frac{1}{2}\frac{\partial^2}{\partial s_1^2} + \ldots\right)b_{k1}(s_1) + \right.\right.$$

$$\left. + \sigma_k\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)b_{k2}(s_1)\right]\exp\frac{\sigma_k n_1}{\lambda} +$$

$$\left. + \sum_{p=1}^{\infty}\frac{1}{H_1}r_p(\zeta)\left[c_{p2}(s_1)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)\right]'_{s_1}\exp\frac{\gamma_p n_1}{\lambda}\right\} + \ldots \qquad (2.5)$$

$$\sigma_z^\circ = 2\mu\lambda\sum_{p=1}^{\infty}t_p(\zeta)\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)c_{p2}(s_1)\exp\frac{\gamma_p n_1}{\lambda} +$$

$$+ 2\mu\lambda^2\sum_{p=1}^{\infty}t_p(\zeta)\left[\left(1 - n_1\frac{a}{2R_1} + n_1^2\frac{3a^2}{8R_1^2} - \ldots\right)c_{p3}(s_1) + \right.$$

$$\left. + \frac{1}{2\gamma_p}\left(-n_1\frac{a^2}{4R_1^2} - n_1\frac{\partial^2}{\partial s_1^2} + \ldots\right)c_{p2}(s_1)\right]\exp\frac{\gamma_p n_1}{\lambda} + \ldots \qquad \left(H_1 = 1 + n_1\frac{a}{R_1}\right) \quad (2.6)$$

Here $R_1(s_1)$ is the radius of curvature of the external contour. All of the remaining notation is explained in [1 and 3]. We recall only that $\psi(s_1, n_1)$ is a certain biharmonic function represented by the series

$$\psi(s_1, n_1) = \psi_0(s_1, n_1) + \lambda\psi_1(s_1, n_1) + \lambda^2\psi_2(s_1, n_1) + \ldots$$

where $\psi_0(s_1, n_1)$ is the solution of the problem of bending of a solid plate given by the applied theory. The functions $\psi_i(s_1, n_1)$, $b_{ki}(s_1)$, and $c_{pi}(s_1)$ are determined by the boundary conditions on Γ_1 from a certain infinite system of linear algebraic equations.

It follows from (2.1) to (2.6) that if n_1 is sufficiently large in absolute value, i.e. at a sufficiently great distance into the plate from Γ_1 (in practice about two to three plate thicknesses), then we need consider only the biharmonic state of stress, and hence we may assume that within the plate the state of stress is determined by Formulas

$$\sigma_x^\circ = 2\mu\lambda\left\{\left[2\nu\frac{\partial^2\psi}{\partial\xi^2} + (\nu-1)\frac{\partial^2\psi}{\partial\eta^2}\right]\zeta - \left(\nu+\frac{1}{3}\right)\lambda^2\frac{\partial^2\triangle\psi}{\partial\xi^2}\frac{\zeta^3}{2}\right\}$$

$$\sigma_y^\circ = 2\mu\lambda\left\{\left[2\nu\frac{\partial^2\psi}{\partial\eta^2} + (\nu-1)\frac{\partial^2\psi}{\partial\xi^2}\right]\zeta - \left(\nu+\frac{1}{3}\right)\lambda^2\frac{\partial^2\triangle\psi}{\partial\eta^2}\frac{\zeta^3}{2}\right\}$$

$$\overset{\circ}{\tau}_{xy} = 2\mu\lambda\left\{(\nu+1)\frac{\partial^2\psi}{\partial\xi\partial\eta}\zeta - \left(\nu+\frac{1}{3}\right)\frac{\partial^2\triangle\psi}{\partial\xi\partial\eta}\frac{\zeta^3}{2}\right\}, \qquad \overset{\circ}{\tau}_{xz} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial\triangle\psi}{\partial\xi}$$

$$\overset{\circ}{\tau}_{yz} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial\triangle\psi}{\partial\eta}, \qquad \sigma_z^\circ = 0 \qquad \left(\xi = \frac{x}{a},\ \eta = \frac{y}{a}\right) \tag{2.7}$$

We note that Formulas (2.7) differ from (1.1) only by terms of higher order in λ.

Since for the time being we have assumed that the plate does not have a hole, the stresses acting on the surface Γ_2, according to (2.7), are

$$\sigma_{n_2}^\circ = 2\mu\lambda\left\{\left[2\nu\frac{\partial^2\psi}{\partial n_2^2} + (\nu-1)\left(\frac{\partial^2\psi}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial\psi}{\partial n_2}\right)\right]\zeta - \left(\nu+\frac{1}{3}\right)\lambda^2\frac{\partial^2\triangle\psi}{\partial n_2^2}\frac{\zeta^3}{2}\right\}_{n_2=0} \tag{2.8}$$

$$\overset{\circ}{\tau}_{n_2s_2} = 2\mu\lambda\left\{\left((\nu+1)\left(\frac{\partial^2\psi}{\partial n_2\partial s_2} - \frac{a}{R_2}\frac{\partial\psi}{\partial s_2}\right)\zeta - \left(\nu+\frac{1}{3}\right)\lambda^2\left(\frac{\partial^2\triangle\psi}{\partial n_2\partial s_2} - \frac{a}{R_2}\frac{\partial\triangle\psi}{\partial s_2}\right)\frac{\zeta^3}{2}\right\}_{n_2=0} \tag{2.9}$$

$$\overset{\circ}{\tau}_{n_2z} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial\triangle\psi}{\partial n_2}\bigg|_{n_2=0} \tag{2.10}$$

where the quantities n_2, s_2 and R_2 refer to Γ_2 (Fig.1).

In order to free Γ_2 of stress it is necessary to remove the stresses (2.8) to (2.10), i.e. to superimpose on (2.7) the state of stress corresponding to the solution of the problem of elasticity theory for an infinite plate with a hole Γ_2, on which the stresses have the values $-\sigma_{n_2}, -\tau_{n_2s_2}, -\tau_{n_2z}$, given by (2.8) to (2.10). It is understood that the state of stress must be compatiable, i.e. the plane boundaries Γ_3 of the plate must remain free of stress and the state of stress must disappear for large n_2. Below we will call this state of stress the reflected state, for short. The reflected state of stress is also derived by the method explained in [1]. Consequently, this state of stress will have the form [1]

$$\sigma_{n_2}^* = 2\mu\lambda\left\{\left[2\nu\frac{\partial^2\psi^*}{\partial n_2^2} + (\nu-1)\left(\frac{1}{H_2^2}\frac{\partial^2\psi^*}{\partial s_2^2} + \frac{1}{H_2}\frac{a}{R_2}\frac{\partial\psi^*}{\partial n_2} + \frac{1}{H_2^3}n_2\frac{aR_2}{R_2^2}\frac{\partial\psi^*}{\partial s_2}\right)\right]\zeta -\right.$$

$$\left.-\left(\nu+\frac{1}{3}\right)\lambda^2\frac{\partial^2\triangle\psi^*}{\partial n_2^2}\frac{\zeta^3}{2}\right\} + 2\mu\lambda\sum_{p=1}^{\infty}\left[(\nu-1)s_p(\zeta) + \gamma_p^2 n_p(\zeta)\right]\left(1 + n_2\frac{a}{2R_2} +\right.$$

$$\left.+ n_2^2\frac{3a^2}{8R_2^2} + \ldots\right)c_{p2}^*(s_2)\exp\left(-\frac{\gamma_p n_2}{\lambda}\right) + 2\mu\lambda^2\left\{2\sum_{k=0}^{\infty}\sin\sigma_k\zeta\,\frac{1}{H_2}\left[b_{k1}^*(s_2)\left(1 + n_2\frac{a}{2R_2} +\right.\right.\right.$$

$$\left.\left.+ n_2^2\frac{3a^2}{8R_2^2} + \ldots\right)\right]_{s_2}\exp\left(-\frac{\sigma_k n_2}{\lambda}\right) + (\nu-1)\sum_{p=1}^{\infty}s_p(\zeta)\left[\left(1 + n_2\frac{a}{2R_2} +\right.\right.$$

$$+ n_2^2\frac{3a^2}{8R_2^2}+\dots\Big) c_{p3}^*(s_2)+\frac{1}{2\gamma_p}\Big(n_2\frac{a^2}{4R_2^2}+n_2\frac{\partial^2}{\partial s_2^2}+\dots\Big) c_{p2}^*(s_2)\Big]\exp\Big(-\frac{\gamma_n n_2}{\lambda}\Big)+$$

$$+\sum_{p=1}^{\infty} n_p(\zeta)\,\gamma_p\Big[\gamma_p\Big(1+n_2\frac{a}{2R_2}+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)c_{p3}^*(s_2)+$$

$$+\Big(-\frac{a}{R_2}-n_2\frac{11a^2}{8R_2^2}+n_2\frac{1}{2}\frac{\partial^2}{\partial s_2^2}+\dots\Big)c_{p2}^*(s_2)\Big]\Big\}\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big)+\dots \qquad (2.11)$$

$$\sigma_{s_2}^* = 2\mu\lambda\Big\{\Big[2\nu\frac{1}{H_2^2}\frac{\partial^2\psi^*}{\partial s_2^2}+\frac{1}{H_2}\frac{a}{R_2}\frac{\partial\psi^*}{\partial n_2}+\frac{1}{H_2^3}n_2\frac{aR_2'}{R_2^2}\frac{\partial\psi^*}{\partial s_2}\Big)+(\nu-1)\frac{\partial^2\psi^*}{\partial n_2^2}\Big]\zeta-$$

$$-\Big(\nu+\frac{1}{3}\Big)\lambda^2\Big(\frac{1}{H_2^2}\frac{\partial^2\triangle\psi^*}{\partial s_2^2}+\frac{1}{H_2}\frac{a}{R_2}\frac{\partial\triangle\psi^*}{\partial n_2}+\frac{1}{H_2^3}n_2\frac{aR_2'}{R_2^2}\frac{\partial\triangle\psi^*}{\partial s_2}\Big)\frac{\zeta^3}{2}\Big\}+$$

$$+2\mu\lambda(\nu-1)\sum_{p=1}^{\infty}s_p(\zeta)\Big(1+n_2\frac{a}{2R_2}+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)c_{p2}^*(s_2)\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big)+$$

$$+2\mu\lambda^2\Big\{-2\sum_{k=0}^{\infty}\sin\sigma_k\zeta\,\frac{1}{H_2}\Big[b_{k1}^*(s_2)\Big(1+n_2\frac{a}{2R_2}+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)\Big]'_{s_2}\exp\Big(-\frac{\sigma_k n_2}{\lambda}\Big)+$$

$$+(\nu-1)\sum_{p=1}^{\infty}s_p(\zeta)\Big[\Big(1+n_2\frac{a}{2R_2}+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)c_{p3}^*(s_2)+\frac{1}{2\gamma_p}\Big(n_2\frac{a^2}{4R_2^2}+$$

$$+n_2^2\frac{\partial^2}{\partial s_2^2}+\dots\Big)c_{p2}^*(s_2)\Big]\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big)+\frac{1}{H_2}\frac{a}{R_2}\sum_{p=1}^{\infty}n_p(\zeta)\,\gamma_p\Big(1+n_2\frac{a}{2R_2}+$$

$$+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)c_{p2}^*(s_2)\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big)\Big\}+\dots \qquad (2.12)$$

$$\tau_{n_2 s_2}^* = 2\mu\lambda\Big\{(\nu+1)\Big(\frac{1}{H_2}\frac{\partial^2\psi^*}{\partial n_2\partial s_2}-\frac{1}{H_2^2}\frac{a}{R_2}\frac{\partial\psi^*}{\partial s_2}\Big)\zeta-\Big(\nu+\frac{1}{3}\Big)\lambda^2\frac{\zeta^3}{2}\Big(\frac{1}{H_2}\frac{\partial^2\triangle\psi^*}{\partial n_2\partial s_2}-$$

$$-\frac{1}{H_2^2}\frac{a}{R_2}\frac{\partial\triangle\psi^*}{\partial s_2}\Big)\Big\}+2\mu\lambda\Big\{-\sum_{k=0}^{\infty}\sigma_k\sin\sigma_k\zeta\Big(1+n_2\frac{a}{2R_2}+$$

$$+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)b_{k1}^*(s_2)\exp\Big(-\frac{\sigma_k n_2}{\lambda}\Big)\Big\}+2\mu\lambda^2\Big\{\sum_{k=0}^{\infty}\sin\sigma_k\zeta\Big[\frac{1}{H_2}\frac{a}{R_2}\Big(1+n_2\frac{a}{2R_2}+$$

$$+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)b_{k1}^*(s_2)-\sigma_k\Big(1+n_2\frac{a}{2R_2}+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)b_{k2}^*(s_2)-$$

$$-\Big(-\frac{a}{R_2}-n_2\frac{11a^2}{8R_2^2}+n_2\frac{1}{2}\frac{\partial^2}{\partial s_2^2}+\dots\Big)b_{k1}^*(s_2)\Big]\exp\Big(-\frac{\sigma_k n_2}{\lambda}\Big)+$$

$$+\frac{1}{H_2}\sum_{p=1}^{\infty}\gamma_p n_p(\zeta)\Big[c_{p2}^*(s_2)\Big(1+n_2\frac{a}{2R_2}+n_2^2\frac{3a^2}{8R_2^2}+\dots\Big)\Big]'_{s_2}\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big)\Big\}+\dots \qquad (2.13)$$

$$\tau^*_{n_2 z} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial\triangle\psi^*}{\partial n_2} + 2\mu\lambda\sum_{p=1}^{\infty}\gamma_p r_p(\zeta)\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)c^*_{p2}(s_2)\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big) + 2\mu\lambda^2\Big\{\sum_{k=0}^{\infty}\frac{1}{H_2}\cos\sigma_k\zeta\Big[b^*_{k1}(s_2)\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)\Big]'_{s_2}\exp\Big(-\frac{\sigma_k n_2}{\lambda}\Big) + \sum_{p=1}^{\infty} r_p(\zeta)\Big[\gamma_p\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)c^*_{p3}(s_2) + \Big(-\frac{a}{2R_2} - n_2\frac{5a^2}{8R_2^2} + n_2\frac{1}{2}\frac{\partial^2}{\partial s_2^2} + \ldots\Big)c^*_{p2}(s_2)\Big]\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big)\Big\} + \ldots \qquad (2.14)$$

$$\tau^*_{s_2 z} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{1}{H_2}\frac{\partial\triangle\psi^*}{\partial s_2} - 2\mu\lambda\sum_{k=0}^{\infty}\sigma_k\cos\sigma_k\zeta\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)b^*_{k1}(s_2)\exp\Big(-\frac{\sigma_k n_2}{\lambda}\Big) + 2\mu\lambda^2\Big\{-\sum_{k=0}^{\infty}\cos\sigma_k\zeta\Big[\Big(-\frac{a}{2R_2} - n_2\frac{5a^2}{8R_2^2} + n_2\frac{1}{2}\frac{\partial^2}{\partial s_2^2} + \ldots\Big)b^*_{k1}(s_2) + \sigma_k\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)b^*_{k2}(s_2)\Big]\exp\Big(-\frac{\sigma_k n_2}{\lambda}\Big) + \sum_{p=1}^{\infty}\frac{1}{H_2}r_p(\zeta)\Big[c^*_{p2}(s_2)\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)\Big]'_{s_2}\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big)\Big\} + \ldots \qquad (2.15)$$

$$\sigma_z^* = 2\mu\lambda\sum_{p=1}^{\infty}t_p(\zeta)\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)c^*_{p2}(s_2)\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big) + 2\mu\lambda^2\sum_{p=1}^{\infty}t_p(\zeta)\Big[\Big(1 + n_2\frac{a}{2R_2} + n_2^2\frac{3a^2}{8R_2^2} + \ldots\Big)c^*_{p3}(s_2) + \frac{1}{2\gamma_p}\Big(n_2\frac{a^2}{4R_2^2} + n_2\frac{\partial^2}{\partial s_2^2} + \ldots\Big)c^*_{p2}(s_2)\Big]\exp\Big(-\frac{\gamma_p n_2}{\lambda}\Big) + \ldots \qquad (2.16)$$

where $\psi^*(s_2, n_2)$ is a biharmonic function having the form

$$\psi^*(s_2, n_2) = \psi_0^*(s_2, n_2) + \lambda\psi_1^*(s_2, n_2) + \lambda^2\psi^*_2(s_2, n_2) + \ldots$$

The boundary values for $\psi_1^*(s_2, n_2)$ and the functions

$$b^*_{ki}(s_2) \quad (i = 1, 2, \ldots),\quad c^*_{pi}(s_2) \quad (i = 2, 3, 4, \ldots)$$

are determined from an infinite system of linear algebraic equations [1]. The boundary conditions for $\psi_0^*(s_2, n_2)$ are given by the relations

$$\frac{2}{3}\Big[2\nu\frac{\partial^2\psi_0^*}{\partial n_2^2} + (\nu - 1)\Big(\frac{\partial^2\psi_0^*}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial\psi_0^*}{\partial n_2}\Big)\Big]_{n_2=0} = \frac{1}{2\mu}M_{11} \qquad (2.17)$$

$$\frac{2}{3}\Big[2\nu\frac{\partial\triangle\psi_0^*}{\partial n_2} + (\nu + 1)\frac{\partial}{\partial s_2}\Big(\frac{\partial^2\psi_0^*}{\partial n_2\partial s_2} - \frac{a}{R_2}\frac{\partial\psi_0^*}{\partial s_2}\Big)\Big]_{n_2=0} = \frac{1}{2\mu}\Big(Q_{02} + \frac{\partial G_{11}}{\partial s_2}\Big) \qquad (2.18)$$

For determining the functions $b^*_{k1}(s_2)$, $c^*_{p2}(s_2)$ we make use of the equations

$$b_{k1}^{*} = \frac{2(-1)^{k+1}}{\sigma_k^3}(\nu+1)\left(\frac{\partial^2\psi_0^*}{\partial s_2\partial n_2} - \frac{a}{R_2}\frac{\partial\psi_0^*}{\partial s_2}\right)_{n_2=0} - \frac{1}{2\mu}T_{k1} \qquad (k = 0, 1, 2, \ldots) \quad (2.19)$$

$$2\nu^2\gamma_p^3\left(\frac{2}{3}\cos^2\gamma_p - 1\right)c_{p2}^* - 4\nu\sum_{\substack{t=1\\ t\neq p}}^{\infty}\frac{\gamma_p^2\gamma_t^2(\cos^2\gamma_p - \cos^2\gamma_t)}{(\gamma_p^2-\gamma_t^2)^2(\gamma_p-\gamma_t)}[(\nu-1)(\gamma_t^2+\gamma_p^2)+$$

$$+2(\nu+1)\gamma_p\gamma_t]\,c_{t2}^* = -\frac{\gamma_p}{2\mu}N_{p1} - 2(\nu-1)\frac{\sin^2\gamma_p}{\gamma_p}\times$$

$$\times\left[2\nu\frac{\partial^2\psi_0^*}{\partial n_2^2} + (\nu-1)\left(\frac{\partial^2\psi_0^*}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial\psi_0^*}{\partial n_2}\right)\right]_{n_2=0} \qquad (p = 1, 2, 3, \ldots) \quad (2.20)$$

In Formulas (2.17) to (2.20)

$$M_{11} = -\frac{4}{3}\mu\left[2\nu\frac{\partial^2\psi_0}{\partial n_2^2} + (\nu-1)\left(\frac{\partial^2\psi_0}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial\psi_0}{\partial n_2}\right)\right]_{n_2=0}, \qquad Q_{02} = -\frac{8}{3}\mu\nu\frac{\partial\triangle\psi_0}{\partial n_2}\bigg|_{n_2=0}$$

$$G_{11} = -\frac{4}{3}\mu(\nu+1)\left(\frac{\partial^2\psi_0}{\partial n_2\partial s_2} - \frac{a}{R_2}\frac{\partial\psi_0}{\partial s_2}\right)_{n_2=0} \qquad (2.21)$$

$$T_{k1} = -4\mu(\nu+1)\frac{(-1)^{k+1}}{\sigma_k^3}\left(\frac{\partial^2\psi_0}{\partial n_2\partial s_2} - \frac{a}{R_2}\frac{\partial\psi_0}{\partial s_2}\right)_{n_2=0}$$

$$N_{p1} = 4\mu(\nu-1)\frac{\sin^2\gamma_p}{\gamma_p^2}\left[2\nu\frac{\partial^2\psi_0}{\partial n_2^2} + (\nu-1)\left(\frac{\partial^2\psi_0}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial\psi_0}{\partial n_2}\right)\right]_{n_2=0} \qquad (2.22)$$

It follows from (2.17), (2.18) and (2.21), if it is noted that the state of stress given by $\psi_0^*(s_2, n_2)$ must vanish at infinity, that $\psi_0^*(s_2, n_2)$ is the solution of the applied theory which frees Γ_2 of the stresses determined by $\psi_0(s_2, n_2)$ and which corresponds to zero stress at infinity. We will call this solution for short, the reflected solution of the applied theory. Furthermore, from (2.17), (2.20) and (2.22) there follows a very important fact: the right-hand side of the system (2.20) reduces to zero in the present case, hence it follows, as was shown in [1], that all $c_{p2}^*(s_2) \equiv 0$, while the $b_{k1}^*(s_2)$, determined from (2.19) are in the general case different from zero. Therefore the state of stress on the contour Γ_2 for $n_2 = 0$ is given by Formulas

$$\sigma_n = \sigma_{n_2}^{\circ} + \sigma_{n_2}^{*} = 2\mu\lambda\left\{\zeta\left[2\nu\frac{\partial^2}{\partial n_2^2} + (\nu-1)\left(\frac{\partial^2}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial}{\partial n_2}\right)\right](\psi+\psi^*) -\right.$$

$$\left.-\left(\nu+\frac{1}{3}\right)\lambda^2\frac{\zeta^3}{2}\frac{\partial^2}{\partial n_2^2}\triangle(\psi+\psi^*)\right\}_{n_2=0} + 2\mu\lambda^2\left\{2\sum_{k=0}^{\infty}\sin\sigma_k\zeta\, b_{k1}^{*\prime}(s_2) +\right.$$

$$\left.+(\nu-1)\sum_{p=1}^{\infty}s_p(\zeta)\,c_{p3}^*(s_2) + \sum_{p=1}^{\infty}n_p(\zeta)\,\gamma_p^2 c_{p3}^*(s_2)\right\} + \ldots = 0 \qquad (2.23)$$

$$\sigma_s = \sigma_{s_2}^{\circ} + \sigma_{s_2}^{*} = 2\mu\lambda\left\{\zeta\left[2\nu\left(\frac{\partial^2}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial}{\partial n_2}\right) + (\nu-1)\frac{\partial^2}{\partial n_2^2}\right](\psi+\psi^*) -\right.$$

$$\left.-\left(\nu+\frac{1}{3}\right)\lambda^2\frac{\zeta^3}{2}\left(\frac{\partial^2}{\partial s_2^2} + \frac{a}{R_2}\frac{\partial}{\partial n_2}\right)\triangle(\psi+\psi^*)\right\}_{n_2=0} +$$

$$+2\mu\lambda^2\left\{-2\sum_{k=0}^{\infty}\sin\sigma_k\zeta\, b_{k1}^{*\prime}(s_2) + (\nu-1)\sum_{p=1}^{\infty}s_p(\zeta)c_{p3}^*(s_2)\right\} + \ldots \qquad (2.24)$$

$$\tau_{ns}=\tau^{\circ}_{n_2s_2}+\tau^{*}_{n_2s_2}=2\mu\lambda\left\{(\nu+1)\,\zeta\left(\frac{\partial^2}{\partial n_2\partial s_2}-\frac{a}{R_2}\frac{\partial}{\partial s_2}\right)(\psi+\psi^*)-\right.$$

$$\left.-\left(\nu+\frac{1}{3}\right)\lambda^2\frac{\zeta^3}{2}\left(\frac{\partial^2}{\partial n_2\partial s_2}-\frac{a}{R_2}\frac{\partial}{\partial s_2}\right)\triangle(\psi+\psi^*)\right\}_{n_2=0}+2\mu\lambda\left\{-\sum_{k=0}^{\infty}\sigma_k\sin\sigma_k\zeta b_{k1}^{*}(s_2)\right\}+$$

$$+2\mu\lambda^2\left\{\sum_{k=0}^{\infty}\sin\sigma_k\zeta\left[\frac{2a}{R_2}\,b_{k1}^{*}(s_2)-\sigma_k b_{k2}^{*}(s_2)\right]\right\}+\ldots=0 \qquad (2.25)$$

$$\tau_{nz}=\tau^{\circ}_{n_2z}+\tau^{*}_{n_2z}=2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial}{\partial n_2}\triangle(\psi+\psi^*)\,|_{n_2=0}+$$

$$+2\mu\lambda^2\left\{\sum_{k=0}^{\infty}\cos\sigma_k\zeta b_{k1}^{*'}(s_2)+\sum_{p=1}^{\infty}\gamma_p r_p(\zeta)\,c_{p3}^{*}(s_2)\right\}+\ldots=0 \qquad (2.26)$$

$$\tau_{sz}=\tau^{\circ}_{s_2z}+\tau^{*}_{s_2z}=2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial}{\partial s_2}\triangle(\psi+\psi^*)\,\Big|_{n_2=0}-$$

$$-2\mu\lambda\sum_{k=0}^{\infty}\sigma_k\cos\sigma_k\,\zeta b_{k1}^{*}(s_2)+2\mu\lambda^2\left\{-\sum_{k=0}^{\infty}\cos\sigma_k\zeta\left[-\frac{a}{2R_2}\,b_{k1}^{*}(s_2)+\sigma_k b_{k2}^{*}(s_2)\right]\right\}+\ldots \qquad (2.27)$$

$$\sigma_z=\sigma_z^{\circ}+\sigma_z^{*}=2\mu\lambda^2\sum_{p=1}^{\infty}t_p(\zeta)\,c_{p3}^{*}(s_2)+\ldots \qquad (2.28)$$

The relation (2.24) shows that the stress σ_s on Γ_2 has the form

$$\sigma_s=\sigma_{s1}\lambda+\sigma_{s2}\lambda^2+\sigma_{s3}\lambda^3+\ldots \qquad (2.29)$$

where the term $\sigma_{s1}\lambda$ corresponds to the solution of the applied theory.

Thus the error in determining the stress σ_s according to the applied theory has at least the next higher order of magnitude in λ than the stress itself. This conclusion is important, since very often the stress concentration coefficient around the hole is determined by the value of σ_s

The stresses τ_{ns} and τ_{nz} in the exact solution are zero on Γ_2. However, the applied theory gives nonzero values for them in the general case. Thus we have here the well-known situation in which the boundary conditions on the tangential stresses are not satisfied. These boundary conditions are satisfied in the applied theory only in the sence of Kirchhoff. Nevertheless, this circumstance still allows the exact asymptotic determination of the stress σ_s, as shown here.

The case of τ_{sz} is somewhat more complicated. From (2.27) we obtain for τ_{sz}

$$\tau_{sz}=2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial}{\partial s_2}\triangle(\psi_0+\psi_0^*)\,\Big|_{n_2=0}-2\mu\lambda\sum_{k=0}^{\infty}\sigma_k\cos\sigma_k\zeta b_{k1}^{*}(s_2)+$$

$$+2\mu\nu\lambda^2\left\{-\sum_{k=0}^{\infty}\cos\sigma_k\zeta\left[-\frac{a}{2R_2}b_{k1}^{*}(s_2)+\sigma_k b_{k2}^{*}(s_2)\right]\right\}+\ldots \qquad (2.30)$$

where the first term corresponds to the solution of the applied theory. From (2.30) it is clear that in the general case τ_{sz} is actually of the first order in λ in the exact solution, while at the same time it is assumed in the applied theory that τ_{sz} is a second order in λ . Thus the applied theory here introduces an error in the order of the quantity considered.

Furthermore, the stress σ_z is equal to zero according to the applied theory, while in reality it is of second order in λ .

From the preceding remarks it follows that if the stress concentration at Γ_2 is determined not on the basis of σ_s , but according to any composite characteristic of the state of stress containing τ_{sz} (for example, the maximum stress), then the use of the applied theory may entail an error of the same order in λ as the quantity itself, characterizing the stress concentration.

We now consider the question in what cases the term of first order in λ in the expression for τ_{sz} vanishes. Obviously the necessary and sufficient condition for this is that all $b_{k1}^*(s_2)$ be equal to zero. From (2.19) it follows that in this case

$$\frac{2(-1)^{k+1}}{\sigma_k^3}(\nu+1)\left(\frac{\partial^2\psi_0^*}{\partial s_2\partial n_2}-\frac{a}{R_2}\frac{\partial\psi_0^*}{\partial s_2}\right)_{n_2=0}=\frac{1}{2\mu}T_{k1}\qquad (k=0,1,2,\ldots)\quad (2.31)$$

Substituting (2.31) into (2.18) and taking note of (2.21) and (2.22), in this case we obtain

$$\frac{4}{3}\nu\frac{\partial\triangle\psi_0^*}{\partial n_2}\bigg|_{n_2=0}=\frac{1}{2\mu}Q_{02}=-\frac{4}{3}\nu\frac{\partial\triangle\psi_0}{\partial n_2}\bigg|_{n_2=0}\qquad (2.32)$$

$$\frac{2}{3}(\nu+1)\left(\frac{\partial^2\psi_0^*}{\partial n_2\partial s_2}-\frac{a}{R_2}\frac{\partial\psi_0^*}{\partial s_2}\right)_{n_2=0}=\frac{1}{2\mu}G_{11}=-\frac{2}{3}(\nu+1)\left(\frac{\partial^2\psi_0}{\partial n_2\partial s_2}-\frac{a}{R_2}\frac{\partial\psi_0}{\partial s_2}\right)_{n_2=0}\quad (2.33)$$

from which it follows that the solution of the applied theory must satisfy the boundary conditions for τ_{ns} and τ_{nz} separately, and not only in the sense of Kirchhoff.

Note that under these conditions $c_{p3}^*(s_2)\equiv 0\ (p=1,2,3,\ldots)$.

The calculation of the stress concentration on the basis of the applied theory for a reinforced hole is very important from the practical point of view.

Very often the calculation of the reinforcing ring is also carried out on the basis of the applied theory of bending of plates [4]. It is clear from the preceding that the applied theory may provide an asymptotically correct value of the concentration coefficient (if it is determined from the stress σ_s) only in the case where the width of the reinforcing ring is several times greater than its thickness. If however the width of the reinforcing ring is comparable to or even less than its thickness, then in that case the edge effects associated with the rotational and potential stress fields in the reinforcing ring will not be damped. The possibility of using the applied theory in that case must be further investigated. In exactly the same way, it is also necessary to consider the case where the state of deformation in the reinforcing ring is described on the basis of Kirchhoff's theory of thin rods.

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